MATH 2050 C Lecture 12 (Feb 25)

[Problem Set 6 posted, due on Mar 5.]

<u>Recall</u>: $(x_n)_{n\in\mathbb{N}} \rightsquigarrow$ subseq. $(x_{n_k})_{k\in\mathbb{N}}$

Thm: Suppose $\lim_{n\to\infty} x_n = x$. Then, every subseq. (x_{n_k}) of (x_n) also converges to the same limit, i.e. $\lim_{k\to\infty} x_{n_k} = x$.

<u>Proof</u>: <u>Note</u>: $n_k \geq k$ for all $k \in \mathbb{N}$ (by induction).

Let $\varepsilon > 0$ be fixed but arbitrary.

$\lim_{n\to\infty} x_n = x \Rightarrow \exists K \in \mathbb{N}$ s.t. $|x_n - x| < \varepsilon \quad \forall n \geq K$

By <u>Note</u> above, when $k \geq K$, then $n_k \geq k \geq K$. Thus,

$$|x_{n_k} - x| < \varepsilon \qquad \forall k \geq K$$

$\square$

<u>Example</u>: Show that $\lim_{n\to\infty} c^{\frac{1}{n}} = 1$ for $c > 1$.

<u>Pf</u>: Let $z_n := c^{\frac{1}{n}}$. Then, by induction,

(z_n) is decreasing and bdd below by 1

By MCT, $\lim_{n\to\infty} (z_n) =: z$ exists.

Consider the subseq. $(z_{n_k})_{k\in\mathbb{N}} := (z_{2k})$, by Thm above,

$$\lim_{k\to\infty} (z_{n_k}) = z.$$

Now, $z_{2n} = c^{\frac{1}{2n}} = (c^{\frac{1}{n}})^{\frac{1}{2}} = (z_n)^{\frac{1}{2}}$

$\because z_n > 1 \ \forall n \in \mathbb{N}$ rejected

Take $n\to\infty$ on both sides, we have $z = \sqrt{z} \Rightarrow z = \cancel{0}$ or 1. $\square$

In summary.

MCT: (x_n) monotone + bdd $\Rightarrow$ (x_n) convergent.

Thm: (x_n) convergent $\underset{\not\Longleftarrow}{\Longrightarrow}$ (x_n) bdd

Thm: (x_n) convergent $\underset{\Longleftarrow\,\checkmark}{\Longrightarrow}$ ANY subseq. (x_{n_k}) of (x_n) converge to the SAME limit.

Take negation yields two divergence criteria:

Cor: (x_n) unbdd $\Rightarrow$ (x_n) divergent

Cor: Either: $\exists$ subseq (x_{n_k}) which is divergent

or : $\exists$ two subseq (x_{n_k}) and $(x_{n_k'})$ s.t

$$\lim_{k\to\infty}(x_{n_k}) \neq \lim_{k\to\infty}(x_{n_k'})$$

$\Longrightarrow$ (x_n) divergent.

Example: $((-1)^n)$ is divergent since $\exists$ two subseq.

$$(1, 1, 1, 1, \dots, 1) \to 1$$

$$(-1, -1, -1, -1, \dots, -1) \to -1$$

Example: $\left(\cos\frac{n\pi}{2}\right) = (0, -1, 0, 1, 0, -1, 0, 1, \dots)$

$\exists$ subseq. $(0, 0, \dots, 0) \to 0$.

$(-1, 1, -1, 1, \dots)$ divergent $\Rightarrow$ original seq is divergent.

Example: $(x_n) = (0, 1, 0, 2, 0, 3, 0, \dots, 0, n, \dots)$ divergent

since $\exists$ subseq. $(1, 2, 3, 4, \dots, n, \dots)$ unbdd $\Rightarrow$ divergent.

Recall: (x_n) divergent $\iff$ (x_n) DOES NOT converge to x for ANY $x \in \mathbb{R}$.

Thm: Fix $x \in \mathbb{R}$. Then

(x_n) does NOT converge to x — either (x_n) divergent, or $(x_n) \to x' \neq x$

$\iff$ $\exists\, \varepsilon_0 > 0$ AND a subseq. (x_{n_k}) of (x_n) s.t.

$$|x_{n_k} - x| \geq \varepsilon_0 \quad \forall k \in \mathbb{N}.$$

Proof: Recall:

$$\lim_{n\to\infty} (x_n) = x \iff \forall \varepsilon > 0,\ \exists K = K(\varepsilon) \in \mathbb{N} \text{ s.t. } |x_n - x| < \varepsilon \quad \forall n \geq K$$

Negate the above.

(x_n) does NOT converge to x $\iff$ $\exists\, \varepsilon_0 > 0$ s.t. $\forall K \in \mathbb{N}$ s.t. $\exists\, n_K \geq K$ s.t. $|x_{n_K} - x| \geq \varepsilon_0$

- Take $K = 1$, choose $n_1 \geq 1$ s.t. $|x_{n_1} - x| \geq \varepsilon_0$
- Take $K = n_1 + 1$, choose $n_2 \geq n_1 + 1$ s.t. $|x_{n_2} - x| \geq \varepsilon_0$

repeat $\rightsquigarrow$ $(x_{n_k})_{k\in\mathbb{N}}$ s.t. $|x_{n_k} - x| \geq \varepsilon_0$

$\blacksquare$